An Introduction to Limits

June 16, 2006

1 Introduction

The purpose of this series of papers is to serve as an introduction to the calculus which acknowledges the existence of modern mathematical concepts and using these concepts to provide an intuitive and easily graspable offering whose nature is not one of seeming arbitrariness. This particular paper serves as an introduction to limits and is fairly self contained - no knowledge beyond what is to be expected of a fifteen year old is supposed. Although in practice, a certain familiarity with set theory and its results proves to be very useful. An important part of these papers is that they will make every attempt to force thinking and contemplation on the part of the reader. More than anything, it is the hope that these series of works aid the reader in increasing their ability with critical thinking - little compares to the freeing of mind such would invoke.

We skim quickly over those topics which are necessary to understand this work. If you already understand these concepts then it is suggested that you simply skip over them else you may find the going dull.

2 A Brief Overview of Limits

This paper attempts to explain the concept of limits rigorously by using hyperreals as a basis for the definition of a limit. This allows for the rigorous use of infinitesimals and as such a rigorously motivated but intuitive grasp of what a limit is. The hyperreals themselves are not built step by step from the naturals nor are the axioms for their legitimacy listed and considered in much detail, only sketched at with particular focus on the transfer principle. The focus of this work is on limits and not the hyppereals. But what is a limit?

In the simplest sense a limit is simply the value some function of some variable (for now we call the result of some equation) approaches as our variable approaches and gets infinitely close to but not equal to some value, we will call c. If we refer to our limit as L, we know that the result of our function $f(x)$ gets infinitely close to L as x approaches c, it is limited to L. L is the limiting value of our function. The limit of a function is often written as $lim\ f$, where lim

is a shortening of limit and f is any function. Thus we may state our original description as:

$$\lim_{x \to c} f(x) = L$$

It is wrong to think of the limit as something that is infinitely approached but never reached, the limit is a realized thing and is something that when manipulated should be treated as a whole and not with particular focus on the parts with the inifinities involved. This only leads to confusion and misconceptions. With the hyperreals we may arrive at a stronger understanding of the concept of limit that is less shaky.

Example 1 *For an example we consider the function* $f(x) = x^3 + 2x$*. The limit of* $f(x)$ *as* x *approaches 5 is 135. To evaluate this we simply substitute 5 for* x *in our equation (since it is the value our variable, x, approaches indefinately) to get:*

$$\lim_{x \to 5} f(x) = 135$$

The notion of the continuity of a function is tied to whether or not a limit may be found, from both directions, on a function. All this and more shall be gone over from the more comfortable setting of hyperreals later. But for now, we must set up the basis of what allows us to use the hyperreals with rigour.

3 A Set

In the most basic sense a set is simply a collection of objects - real or imagined. If you are observant you will notice that I have just defined the word by simply replacing it with a synonym - collection for set - akin to stating that a Bucket is a pail which holds water. This cannot be helped, a set is a fundamental and irreducible concept. In essence a set is an aggregate or grouping of the objects which are its members. For example, a similar concept of an aggregate is to refer to 8 wolves as simply a pack of wolves.

3.1 The Contents and Form of a Set

A naive set is the intuitive concept of a set. It is basically the concept of a set in its raw form, a prototype of a set before its kinks are worked out. Consider a game which has just been thought up. This game must be played and tested to figure out what works and what doesn't, which rules to include or not include. Naive sets are like this, they are the concept of sets used intuitively without regard for the issues or paradoxes that such a concept may involve. Naive sets can have anything for members with any combination. There generally are no rules as to what type of objects can be included in a set or how they are included. So we may have a set:

$$\{a, \textit{all tall people}, \text{“}g\text{”}, 3, 44, \textit{this set}\}$$

with no apparent pattern as to what decides its members. Conventionally (in the western hemisphere) the members of a set are enclosed within two braces: {} with the set itself represented by capitalized bolded roman letters. The members of a set are often represented by variables which are often simply lower case roman letters. Thus for the set with members a,b,c we may write:

$$A \ = \{a, b, c\}$$

The ordering of the members of a set and the number of times the members appear are not important, these are not distinguishing factors. Thus the sets {a,b,c} , {,a,a,a,b,b,b,b,c,c} and {b,b,b,a,c,c} are all equivalent since they all have the same members, regardless of order or number of times written. We will return to the concept of membership later.

3.2 Russell's Paradox

A set may also contain itself as a member. So we can have $D = \{a, b, c, D\}$. Now, consider a set $S = \{$*Any set which does not have itself as a member*$\}$. So we know that D is not a member of the set S since it has itself as a member. But is S itself a member of this set? Is it a member of itself? Can we have $\{..., S\}$? If S is a member of itself then obviously something is wrong since the requirement of membership is a set must not have itself as a member to be included. But then if we say it is not a member of itself then there is still a problem. Why? Because then it would meet our requirement of a set that does not include itself as a member. Therefore it would in fact be a member of itself! This is obviously a paradox and is known as **Russell's paradox,** named after Bertrand Russell, the mathematician who thought up the paradox around 1901.

To take care of these kind of paradoxes we come up with rules or formally, axioms whose criteria must be met in order for some random entity or object be considered a set. Thus under these axioms we find that our set S does not meet the criteria required to be considered a set. Hence not all collections of objects may be considered a set. Another workaround is to consider hierarchy of sets whose construction are such that at each level in the hierarchy, we could not encounter a set that is a member of itself. With respect to our set S, we must go up at least a level from its members to encounter it.

All these though are solutions to problems offered to make the system which we are working with more consistent and free from problems. Mathematics in this sense is like a patchwork quilt. We sew, sew and if we sew in such a way as to tear our quilt we must patch it up lest the poor person who uses our quilt freeze to death. There are many implications to this paradox and the assumptions required to resolve it (assumption of axiom of choice or well ordering for hierarchy) or ability to prove no strong formal mathematical theory may ever be complete but these are outside the scope of this paper.

3.3 Membership

To denote that a variable is a member of some set we use the symbol: $\in$. $a \in A$, reads a *is an element* of the set A. A set may have subsets. A subset is simply a focusing on certain aspect or members of a set. For example, the set of all females is the subset of the set of humans. The symbol $\subseteq$ means *is a subset of.* Thus if $A = \{a, b, c, d\}$ and the set $B = \{a, b\}$ then $B \subseteq A$. Two sets which have the same members are the same, obvious but worth stating. This is know as the Principle of Extension.

All sets have a subset, known as the null or empty set. The empty set represented by $\varnothing$ is the set which has no members. Consider some set A. To show that $\varnothing$ is a subset of A we must show that all the members of $\varnothing$ are also members of A. Since the empty set has no members it automatically satisfies this criteria for all sets, we dont have to show anything. In other words, all *zero* members of $\varnothing$ are certainly members of A, thus $\varnothing$ is the subset of any set A. Sneaky but logical. There is also only one empty set. Since all sets which have no members have the same members, which is nothing, they are the same as the empty set.

A set may have itself be a subset of itself. For example, the set of all humans can be considered to be a subset of the set of all humans. In essence we are still focusing on the entire set itself. In fact all sets have themselves and the empty sets as subsets. Notice that if we have two sets A and B then:

$$A = B \Leftrightarrow (A \subseteq B) \wedge (B \subseteq A)$$

This basically reads: A is *equal* to B *if and only if* (iff) A *is a subset* of B and B *is a subset* of A. This is basically the same as saying that A is a subset of A or $A \subseteq A$.

Example 2 *Consider the set $A = \{1, 2, 3\}$ and some set B. If we have that $B \subseteq A$ then we know that B is either {∅},{1}, {2}, {3}, {1,2}, {1,3}, {2,3} or {1,2,3}. If we have that A is also a subset of B then we know that A can only be the subset of B which is equal to A, the subset of B that is {1,2,3}. Since we have originally that B is a subset of A, we know that B does not have any members that A does not have. Thus B must also be {1,2,3} or A itself. So in other words and in a round about manner we see that A is the same as B.*

There also exists the notion of a proper subset, symbolized with $\subset$. A proper subset is a subset relation which does not allow a set to be a subset of itself. Thus the set of all books is not a *proper subset* of the set of all books while the set of all blue books is a *proper subset* of the set of all books. Unlike $A \subseteq A$ which is true, $A \subset A$ is false. It is interesting to realize that everything is a set (although as is always the case there is an exception with some set theories, see urelements for more). This follows from the fact that there is only one empty set, thus anything which has no members is equivalent to the empty set by extension. Thus everthing is either a set or it is the empty set. This is an important fact which allows us to build or base the enitirety of mathematics on set thoery.

4 Relations

5 Functions

6 Numbers and Number Systems

7 Hyperreals

8 Hyperreal Limits